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STATISTICAL LIMIT POINT AND STATISTICAL CLUSTER POINT IN METRIC-LIKE SPACES

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Abstract. Motivated by the work of Matthews [16] and Harandi [2] and following the line of Fridy [10] and Salat [18], in this paper we introduce and study the notions of statistical limit point and statistical cluster point of sequences in a metric-like space. We study some basic properties of the set of all statistical limit points and the set of all statistical cluster points of a sequence in a metric-like space including their interrelationship.

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1. Introduction and Background

In a metric space $(\mathcal{X}, d)$ [8], the concept of equality ($x = y$) and indistancy ($d(x, y) = 0$) are equivalent and such identification is so natural that any other suggestion seems to serve no purpose. But in some practical problems, particularly in computer science, some times "equality ⇒ indistancy" dose not hold (see [4]). To handle such situation, in 1994 Matthews [16] introduced the idea of partial metric space as an extension of the idea of metric space. For more primary information about partial metric space, researchers can see [1, 4]. In [2], A. A. Harandi introduced the idea of *metric-like space* as an extension of *partial metric space* and introduced the idea of convergence and Cauchyness in a *metric-like space*.

On the other hand, the usual notion of convergence of sequences does not capture a large class of sequences which are not convergent in usual sense. To consider more sequences under purview the usual concept of convergence of real sequences was extended to statistical convergence by Fast [7] (and also independently by Schoenberg [20]) using natural density and it has become one of the most active research area in summability theory after the work of Salat [18] and Fridy [9]. Using the idea of statistical convergence [9], the concept of statistical limit points and statistical cluster points of real sequences was introduced by Fridy [10]. A lot of work has been done in this connection. Interested researchers can see [3, 5, 6, 11, 12, 13, 17, 19] etc.

Recently following Harandi [2] and Salat [18], in [15] Malik et al. have introduced the idea of statistical convergence of sequences in a metric-like space and some basic properties have been studied, which naturally generalize the concepts of convergence of sequences in metric spaces, partial metric spaces and metric-like spaces. It seems therefore reasonable to introduce and study the concepts of statistical limit points and statistical cluster points in a *metric-like space*. In this paper we do the same

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and investigate some basic properties of the set of all statistical limit points and the set of all statistical cluster points of a sequence of points in a metric-like space including their interrelationship. Our results extends the related results in [10, 14].

2. **Basic Definitions and Notations**

Let us begin this section with few basic definitions and notations which will be needed in our study. Throughout the paper, by $\mathbb{R}$ we denote the set of all real numbers and by $\mathbb{R}_{\geq 0}$ we denote the collection of all non-negative real numbers.

First we recall the idea of natural density, statistical convergence, statistical limit points and statistical cluster points of sequences in $\mathbb{R}$.

Definition 2.1. [9] Let $\mathcal{P}$ be a subset of $\mathbb{N}$. For each $l \in \mathbb{N}$, let $\mathcal{P}(l)$ denote the cardinality of the set $\{j \leq l : j \in \mathcal{P}\}$. Then $d(\mathcal{P})$ is said to be the natural density or simply density of the set $\mathcal{P}$, if the limit $\lim\limits_{l\to\infty} \mathcal{P}(l)/l$ exists finitely and

$$d(\mathcal{P}) = \lim_{l\to\infty} \frac{\mathcal{P}(l)}{l}.$$

Definition 2.2. [9] Let $\{x_n\}_{n\in\mathbb{N}} \in \mathbb{R}$. Then $\{x_n\}_{n\in\mathbb{N}}$ is called statistically convergent to $x_o(\in \mathbb{R})$, if $\forall\ \epsilon > 0,\ d(\mathcal{A}(\epsilon)) = 0$, where

$$\mathcal{A}(\epsilon) = \{j \in \mathbb{N} : |x_j - x_o| \geq \epsilon\}.$$

We denote it by $stat - \lim\limits_{n\to\infty} x_n = x_o$.

Definition 2.3. [10] Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence of real numbers. A subsequence $\{x_{n_p}\}_{p\in\mathbb{N}}$ of $\{x_n\}_{n\in\mathbb{N}}$ is called a non-thin subsequence if $d(\{n_1 < n_2 < \ldots\}) \neq 0$, where $d(\{n_1 < n_2 < \ldots\}) \neq 0$ means either it is positive or it does not exist. On the other hand, if $d(\{n_1 < n_2 < \ldots\}) = 0$, $\{x_{n_p}\}_{p\in\mathbb{N}}$ is called a thin subsequence of $\{x_n\}_{n\in\mathbb{N}}$.

Definition 2.4. [10] Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence of points in $\mathbb{R}$ and x_o be a real number. Then x_o is said to be a statistical limit point of $\{x_n\}_{n\in\mathbb{N}}$ if there exists a non-thin subsequence of $\{x_n\}_{n\in\mathbb{N}}$ that converges to x_o.

Definition 2.5. [10] Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence of points in $\mathbb{R}$ and x_o be a real number. Then x_o is said to be a statistical cluster point of $\{x_n\}_{n\in\mathbb{N}}$ if for all $\epsilon > 0$,

$$d(\{n \in \mathbb{N} : |x_n - x_o| < \epsilon\}) \neq 0.$$

We now recall the idea of metric-like spaces.

Definition 2.6. [16] Let $\mathcal{Y}$ be a non-empty set. A partial-metric p on $\mathcal{Y}$ is a mapping $p : \mathcal{Y} \times \mathcal{Y} \to \mathbb{R}_{\geq 0}$ such that for any $u, v, w \in \mathcal{Y}$, the following conditions are satisfied:

$(p1)$ $u = v$ if and only if $p(u, u) = p(u, v) = p(v, v)$;
$(p2)$ $0 \leq p(u, u) \leq p(u, v)$;
$(p3)$ $p(u, v) = p(v, u)$;
$(p4)$ $p(u, v) \leq p(u, w) + p(w, v) - p(w, w)$.

Then $(\mathcal{Y}, p)$ denotes a partial metric space.

Definition 2.7. [2] Let $\mathcal{Y}$ be a non-empty set. A metric-like δ on $\mathcal{Y}$ is a mapping $\delta : \mathcal{Y} \times \mathcal{Y} \to \mathbb{R}_{\geq 0}$ such that for any $u, v, w \in \mathcal{Y}$, it satisfied the conditions $(\delta 1)$ to $(\delta 3)$:

$(\delta 1)$ $\delta(u, v) = 0 \Rightarrow u = v$;

($\delta2$) $\delta(u,v)=\delta(v,u)$;
($\delta3$) $\delta(u,v)\leq\delta(u,w)+\delta(w,v)$.

The pair $(\mathcal{Y},\delta)$ is called a *metric-like space*. Note that in a metric-like space $(\mathcal{Y},\delta)$, $\delta(x,x)$ may be positive for some $x\in\mathcal{Y}$. Throughout the paper, $(\mathcal{Y},\delta)$ will denote a metric-like space, unless otherwise mentioned.

Remark 2.1. From the above two definitions it is clear that if σ is a metric on a non-empty set $\mathcal{Y}$, then σ satisfies all the conditions ($p1$) to ($p4$) of Definition 2.6 and if p is a partial-metric on $\mathcal{Y}$, then it satisfies all the conditions ($\delta1$) to ($\delta3$) of Definition 2.7. But the converses are not true, as we see in the following examples.

Example 2.1. *Let $\mathcal{Y}$ be the set of all real numbers and $p:\mathcal{Y}\times\mathcal{Y}\to\mathbb{R}_{\geq 0}$ be a mapping defined as follows:*

$$p(u,v)=|u-v|+1,\ u,v\in\mathcal{Y}.$$

Then p satisfies all the conditions ($p1$) to ($p4$) of the Definition 2.6. Hence $(\mathcal{Y},p)$ is a partial-metric space. But $p(1,1)=1\neq 0$. Therefore p is not a metric on $\mathcal{Y}$.

Example 2.2. *Let $\mathcal{Y}$ be the set of all non-negative real numbers and a mapping $\delta:\mathcal{Y}\times\mathcal{Y}\to\mathbb{R}_{\geq 0}$ be defined as follows:*

$$\delta(u,v)=\begin{cases}0, & \text{if } u=v \text{ and } u \text{ is an irrational in } \mathcal{Y}\\ u+v+1, & \text{otherwise.}\end{cases}$$

Then δ satisfies all the conditions ($\delta1$) to ($\delta3$) of the Definition 2.7. So $(\mathcal{Y},\delta)$ is a metric-like space. But we see that $\delta(4,4)=9\nleq\delta(4,2)=7$. So δ is not a partial metric on $\mathcal{Y}$.

Definition 2.8. [2] Let x_o be a point in a metric-like space $(\mathcal{Y},\delta)$ and let $\epsilon>0$. Then the open δ-ball with centered at x_o and radius $\epsilon>0$ in $(\mathcal{Y},\delta)$ is denoted by $\mathcal{B}_\delta(x_o;\epsilon)$ and is defined by

$$\mathcal{B}_\delta(x_o;\epsilon)=\{x\in\mathcal{Y}:|\delta(x,x_o)-\delta(x_o,x_o)|<\epsilon\}.$$

Definition 2.9. [2] A sequence $\{x_n\}_{n\in\mathbb{N}}$ of points in a metric-like space $(\mathcal{Y},\delta)$ is called convergent to $x_o\in\mathcal{Y}$, if for every $\epsilon>0$, there exists $k_o\in\mathbb{N}$ such that

$$|\delta(x_n,x_o)-\delta(x_o,x_o)|<\epsilon,\qquad \forall\ n\geq k_o$$
$$\text{i.e. } x_n\in\mathcal{B}_\delta(x_o;\epsilon),\qquad \forall\ n\geq k_o.$$

In this case, we write $\lim\limits_{n\to\infty}x_n=x_o$.

Following [15] we now recall the idea of statistical convergence of a sequence $\{x_p\}_{p\in\mathbb{N}}$ in a metric-like space $(\mathcal{Y},\delta)$.

Definition 2.10. [15] Let $(\mathcal{Y},\delta)$ be a metric-like space and let $\{x_p\}_{p\in\mathbb{N}}$ be a sequence of points in $\mathcal{Y}$. Then $\{x_p\}_{p\in\mathbb{N}}$ is said to convergent statistically to a point x_o ($\in\mathcal{Y}$) if $stat-\lim\limits_{p\to\infty}\delta(x_p,x_o)=\delta(x_o,x_o)$ i.e. for every $\epsilon>0$,

$$d(\{p\in\mathbb{N}:|\delta(x_p,x_o)-\delta(x_o,x_o)|\geq\epsilon\})=0$$
$$\text{i.e. } d(\{p\in\mathbb{N}:x_p\notin\mathcal{B}_\delta(x_o;\epsilon)\})=0.$$

We denote it by $stat-\lim\limits_{p\to\infty}x_p=x_o$.

Theorem 2.1. [15] *Let $(\mathcal{Y}, \delta)$ be a metric-like space, $\{x_n\}_{n\in\mathbb{N}}$ be a sequence of points in $\mathcal{Y}$ and $x_o \in \mathcal{Y}$. Then $stat - \lim_{n\to\infty} x_n = x_o$ if and only if there exists a subsequence $\{x_{n_p}\}_{p\in\mathbb{N}}$ of $\{x_n\}_{n\in\mathbb{N}}$ such that $d(\{n_1 < n_2 < \ldots\}) = 1$ and $\lim_{p\to\infty} x_{n_p} = x_o$.*

3. Statistical Limit point and Statistical Cluster point

We now present the formal definition of limit points of a sequence of points in a metric-like space.

Definition 3.1. Let $(\mathcal{Y}, \delta)$ be a metric-like space and $\{x_n\}_{n\in\mathbb{N}}$ be a sequence of points in $\mathcal{Y}$. A point x_o $(\in \mathcal{Y})$ is called a limit point of $\{x_n\}_{n\in\mathbb{N}}$ if every open δ-ball $\mathcal{B}_\delta(x_o; r)$ $(r > 0)$ contains infinitely many points of $\{x_n\}_{n\in\mathbb{N}}$.

The set of all limit points of a sequence $x = \{x_n\}_{n\in\mathbb{N}}$ in a metric-like space $(\mathcal{Y}, \delta)$ is denoted by $\mathcal{L}(x)$.

Theorem 3.1. *Assume that $(\mathcal{Y}, \delta)$ be a metric-like space and $\{x_n\}_{n\in\mathbb{N}}$ be a sequence of points in $\mathcal{Y}$. A point $x_o \in \mathcal{Y}$ is a limit point of $\{x_n\}_{n\in\mathbb{N}}$ if and only if there exists a subsequence of $\{x_n\}_{n\in\mathbb{N}}$ which converges to x_o.*

Proof. Let x_o be a limit point of $\{x_n\}_{n\in\mathbb{N}}$ in $\mathcal{Y}$. Then for all $l \in \mathbb{N}$, $\mathcal{B}_\delta(x_o; \frac{1}{l})$ contains infinitely many points of $\{x_n\}_{n\in\mathbb{N}}$. So we get a subsequence $\{x_{n_l}\}_{l\in\mathbb{N}}$ of $\{x_n\}_{n\in\mathbb{N}}$ so that

$$x_{n_l} \in \mathcal{B}_\delta(x_o; \frac{1}{l}),\ \forall\ l \in \mathbb{N}. \tag{3.1}$$

Let $\epsilon > 0$ be given. Then there exists $l_o \in \mathbb{N}$ such that $\frac{1}{l_o} < \epsilon$. Then from (3.1) we have,

$$|\delta(x_{n_l}, x_o) - \delta(x_o, x_o)| < \frac{1}{l} \leq \frac{1}{l_o} < \epsilon,\ \forall\ l \geq l_o$$
$$\Rightarrow \lim_{l\to\infty} \delta(x_{n_l}, x_o) = \delta(x_o, x_o).$$

Therefore, the subsequence $\{x_{n_l}\}_{l\in\mathbb{N}}$ converges to x_o.

Conversely, let there exists a subsequence $\{x_{n_l}\}_{l\in\mathbb{N}}$ of $\{x_n\}_{n\in\mathbb{N}}$ such that $\lim_{l\to\infty} x_{n_l} = x_o$. Let $\epsilon > 0$ be given. Then there exists $m_o \in \mathbb{N}$ such that

$$|\delta(x_{n_l}, x_o) - \delta(x_o, x_o)| < \epsilon,\ \forall\ l \geq m_o$$
$$\Rightarrow x_{n_l} \in \mathcal{B}_\delta(x_o; \epsilon),\ \forall\ l \geq m_o.$$

This implies that the open δ-ball $\mathcal{B}_\delta(x_o; \epsilon)$ contains infinitely many points of $\{x_n\}_{n\in\mathbb{N}}$. Hence x_o must be a limit point of $\{x_n\}_{n\in\mathbb{N}}$ in $\mathcal{Y}$. □

Definition 3.2. Let $(\mathcal{Y}, \delta)$ be a metric-like space and $\{x_n\}_{n\in\mathbb{N}}$ be a sequence of points in $\mathcal{Y}$. A point $x_o \in \mathcal{Y}$ is called a statistical limit point of $\{x_n\}_{n\in\mathbb{N}}$ if there exists a non-thin subsequence of $\{x_n\}_{n\in\mathbb{N}}$ which converges to x_o.

The set of all statistical limit points of $x = \{x_n\}_{n\in\mathbb{N}}$ in $(\mathcal{Y}, \delta)$ is denoted by $\Lambda(x)$.

Theorem 3.2. *Let $(\mathcal{Y}, \delta)$ be a metric-like space and $x = \{x_n\}_{n\in\mathbb{N}}$ be a sequence of points in $\mathcal{Y}$. If $stat - \lim_{n\to\infty} x_n = x_o$ in $(\mathcal{Y}, \delta)$, then $x_o \in \Lambda(x)$.*

Proof. Let us assume $x = \{x_n\}_{n\in\mathbb{N}} \in \mathcal{Y}$ and $stat - \lim_{n\to\infty} x_n = x_o$. Then using Theorem 2.1 we have, there must exists a subsequence, say $\{x_{n_p}\}_{p\in\mathbb{N}}$ of $\{x_n\}_{n\in\mathbb{N}}$ such that $d(\{n_1 < n_2 < \ldots\}) = 1$ and $\lim_{p\to\infty} x_{n_p} = x_o$.

Now, $d(\{n_1 < n_2 < \ldots\}) = 1$ implies that $\{x_{n_p}\}_{p\in\mathbb{N}}$ is a non-thin subsequence of $\{x_n\}_{n\in\mathbb{N}}$ which converges to x_o. So, $x_o \in \Lambda(x)$. □

Remark 3.1. The converse of Theorem 3.2 is not true. To show this we consider the following example.

Example 3.1. *Let $\mathcal{Y}$ be the collection of all non-negative real numbers and $\delta : \mathcal{Y}\times\mathcal{Y} \to \mathbb{R}_{\geq 0}$ be defined as follows:*

$$\delta(u,v) = \begin{cases} 0, & \text{if } u = v \text{ and } u \text{ is irrational in } \mathcal{Y} \\ u+v, & \text{otherwise.} \end{cases}$$

Then $(\mathcal{Y},\delta)$ is a metric-like space but not a metric space. We now consider a sequence $\{x_n\}_{n\in\mathbb{N}}$ in $\mathcal{Y}$ as follows:

$$x_n = \begin{cases} \frac{1}{n}, & \text{if } n \text{ is odd} \\ n, & \text{if } n \text{ is even.} \end{cases}$$

Let $\epsilon > 0$ be given. Then there exists $n_o \in \mathbb{N}$ such that $\frac{1}{n_o} < \epsilon$. So,

$$|\delta(x_n,0) - \delta(0,0)| = \frac{1}{n} \leq \frac{1}{n_o} < \epsilon, \ \forall\ n \geq n_o \ \text{ and } n \text{ is odd}$$
$$\Rightarrow \lim_{k\to\infty} \delta(x_{2k-1},0) = \delta(0,0).$$

Therefore, the subsequence $\{x_{2k-1}\}_{k\in\mathbb{N}}$ converges to 0. Since, $d(\{2k-1 : k \in \mathbb{N}\}) = \frac{1}{2} \neq 0$, so $\{x_{2k-1}\}_{k\in\mathbb{N}}$ is a non-thin subsequence of $\{x_n\}_{n\in\mathbb{N}}$ converging to 0 in $\mathcal{Y}$. Therefore $0 \in \Lambda(x_n)$.

But we see that $\{n \in \mathbb{N} : |\delta(x_n,0) - \delta(0,0)| \geq 1\} = \{2k : k \in \mathbb{N}\} \cup \{1\}$. As $d(\{2k : k \in \mathbb{N}\}) = \frac{1}{2}$, so $d(\{n \in \mathbb{N} : |\delta(x_n,0) - \delta(0,0)| \geq 1\}) = \frac{1}{2}$. Therefore, the sequence $\{x_n\}_{n\in\mathbb{N}}$ is not convergent statistically to 0 in $(\mathcal{Y},\delta)$.

Definition 3.3. Let $(\mathcal{Y},\delta)$ be a metric-like space and $\{x_n\}_{n\in\mathbb{N}}$ be a sequence of points in $\mathcal{Y}$. A point $x_o \in \mathcal{Y}$ is called a statistical cluster point of $\{x_n\}_{n\in\mathbb{N}}$ if, for every $\epsilon > 0$,

$$d(\{n \in \mathbb{N} : |\delta(x_n,x_o) - \delta(x_o,x_o)| < \epsilon\}) \neq 0$$
$$\text{i.e. } d(\{n \in \mathbb{N} : x_n \in \mathcal{B}_\delta(x_o;\epsilon)\}) \neq 0.$$

The set of all statistical cluster points of $x = \{x_n\}_{n\in\mathbb{N}}$ in $\mathcal{Y}$ is denoted by $\Gamma(x)$.

Theorem 3.3. *Let $(\mathcal{Y},\delta)$ be a metric-like space and $x = \{x_n\}_{n\in\mathbb{N}}$ be a sequence of points in $\mathcal{Y}$. Then $\Gamma(x) \subset \mathcal{L}(x)$.*

Proof. If $\Gamma(x) = \emptyset$, then the proof is trivial. Let $\Gamma(x) \neq \emptyset$ and $x_o \in \Gamma(x)$. Then for every $p \in \mathbb{N}$,

$$d(\{n \in \mathbb{N} : |\delta(x_n,x_o) - \delta(x_o,x_o)| < \frac{1}{p}\}) \neq 0$$
$$\text{i.e. } A_p = \{n \in \mathbb{N} : |\delta(x_n,x_o) - \delta(x_o,x_o)| < \frac{1}{p}\} \text{ is an infinite set.}$$

Let n_1 be the least element of A_1. As A_2 is an infinite set, so there exists $n_2 \in A_2$ such that $n_1 < n_2$. Similarly there exists $n_3 \in A_3$ such that $n_1 < n_2 < n_3$. Proceeding in this way we will obtained a strictly increasing sequence $\{n_p\}_{p\in\mathbb{N}}$ of natural numbers such that $\{x_{n_p}\}_{p\in\mathbb{N}}$ is a subsequence of $\{x_n\}_{n\in\mathbb{N}}$ and

$$\big|\delta(x_{n_p}, x_o) - \delta(x_o, x_o)\big| < \frac{1}{p},\ \forall\, p \in \mathbb{N}. \tag{3.2}$$

Let $\epsilon > 0$ be given. Then there exists $p_o \in \mathbb{N}$ so that $\frac{1}{p_o} < \epsilon$. So from (3.2) we have,

$$\begin{aligned} &\big|\delta(x_{n_p}, x_o) - \delta(x_o, x_o)\big| < \frac{1}{p} \leq \frac{1}{p_o} < \epsilon,\ \forall\, p \geq p_o \\ \Rightarrow &\lim_{p\to\infty} \delta(x_{n_p}, x_o) = \delta(x_o, x_o). \end{aligned}$$

Therefore, the subsequence $\{x_{n_p}\}_{p\in\mathbb{N}}$ converges to x_o. So, from Theorem 3.1 we have $x_o \in \mathcal{L}(x)$. Hence $\Gamma(x) \subset \mathcal{L}(x)$. □

Remark 3.2. The inclusion in the Theorem 3.3 may be proper. To show this we consider the following example.

Example 3.2. *We consider the metric-like space* $(\mathcal{Y}, \delta)$ *as described in Example 3.1. We consider a sequence* $x = \{x_n\}_{n\in\mathbb{N}}$ *in* $\mathcal{Y}$ *defined as follows:*

$$x_n = \begin{cases} \frac{1}{p}, & \text{if } n = p^2\ (p = 1, 2, 3, \ldots) \\ n, & \text{otherwise.} \end{cases}$$

Let $\epsilon > 0$ *be given. Then there exists* $p_o \in \mathbb{N}$ *such that* $\frac{1}{p_o} < \epsilon$. *So we have,*

$$\begin{aligned} &\big|\delta(x_{p^2}, 0) - \delta(0, 0)\big| = \frac{1}{p} \leq \frac{1}{p_o} < \epsilon,\ \forall\, p \geq p_o \\ \Rightarrow &\lim_{p\to\infty} \delta(x_{p^2}, 0) = \delta(0, 0) \\ \Rightarrow &\lim_{p\to\infty} x_{p^2} = 0. \end{aligned}$$

Therefore, the subsequence $\{x_{p^2}\}_{p\in\mathbb{N}}$ *converges to 0 and so* $0 \in \mathcal{L}(x)$. *But we see that* $\{n \in \mathbb{N} : |\delta(x_n, 0) - \delta(0, 0)| < 1\} = \{2^2, 3^2, 4^2, \ldots\}$. *As* $d(\{2^2, 3^2, 4^2, \ldots\}) = 0$ *so,* $d(\{n \in \mathbb{N} : |\delta(x_n, 0) - \delta(0, 0)| < 1\}) = 0$. *Thus we have,* $0 \notin \Gamma(x)$ *and hence* $\Gamma(x) \subsetneq \mathcal{L}(x)$.

We now formulate the relation between $\Lambda(x)$ and $\Gamma(x)$ for a sequence $\{x_n\}_{n\in\mathbb{N}}$ in a metric-like space $(\mathcal{Y}, \delta)$.

Theorem 3.4. *Let* $(\mathcal{Y}, \delta)$ *be a metric-like space and* $x = \{x_n\}_{n\in\mathbb{N}}$ *be a sequence of points in* $\mathcal{Y}$. *Then* $\Lambda(x) \subset \Gamma(x) \subset \mathcal{L}(x)$.

Proof. Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence of points in $\mathcal{Y}$. If $\Lambda(x) = \emptyset$, then the proof is obvious. Let $\Lambda(x_n) \neq \emptyset$ and $x_o \in \Lambda(x_n)$. Then there exists a non-thin subsequence $\{x_{n_p}\}_{p\in\mathbb{N}}$ of $\{x_n\}_{n\in\mathbb{N}}$ such that $\lim_{p\to\infty} x_{n_p} = x_o$. So,

$$\lim_{p\to\infty} \delta(x_{n_p}, x_o) = \delta(x_o, x_o).$$

Let $\epsilon > 0$ be given. Then there exists $p_o \in \mathbb{N}$ so that

$$\big|\delta(x_{n_p}, x_o) - \delta(x_o, x_o)\big| < \epsilon,\ \forall\, p \geq p_o. \tag{3.3}$$

Let $\mathcal{A}(\epsilon) = \{n \in \mathbb{N} : |\delta(x_n, x_o) - \delta(x_o, x_o)| < \epsilon\}$. Then from (3.3) we can say $\{n_p \in \mathbb{N} : p \geq p_o\} \subset \mathcal{A}(\epsilon)$. As $d(\{n_1 < n_2 < \ldots\}) \neq 0$ so, $d(\{n_p \in \mathbb{N} : p \geq p_o\}) \neq 0$.

So we have $d(\mathcal{A}(\epsilon)) \neq 0$. This implies $x_o \in \Gamma(x)$. Thus we have $\Lambda(x) \subset \Gamma(x)$. Therefore using Theorem 3.3 we have

$$\Lambda(x) \subset \Gamma(x) \subset \mathcal{L}(x).$$

□

Remark 3.3. We now cite an example to claim that the inclusion in the Theorem 3.4 may be proper.

Example 3.3. *We consider the metric-like space $(\mathcal{Y}, \delta)$ as described in Example 3.1.*

Now for each $j \in \mathbb{N}$, let $\mathcal{D}_j = \{2^{j-1}(2s-1) : s \in \mathbb{N}\}$. Then $\mathbb{N} = \bigcup_{j=1}^{\infty} \mathcal{D}_j$, is a decomposition of $\mathbb{N}$ and $\mathcal{D}_i \cap \mathcal{D}_j = \emptyset$ for $i \neq j$. Then following [10] *we have $d(\mathcal{D}_j) = \frac{1}{2^j}$ for each $j \in \mathbb{N}$. Now we define a sequence $x = \{x_n\}_{n\in\mathbb{N}}$ in $\mathcal{Y}$ as follows:*

$$x_n = \frac{1}{j}, \text{ if } n \in \mathcal{D}_j.$$

Fix $j \in \mathbb{N}$ and set $\mathcal{D}_j = \{n_1^{(j)} < n_2^{(j)} < n_3^{(j)} < \ldots\}$. As $d(\mathcal{D}_j) = \frac{1}{2^j}$, we have $\{x_{n_k^{(j)}}\}_{k\in\mathbb{N}}$ is a non-thin subsequence of $\{x_n\}_{n\in\mathbb{N}}$. Now let $\epsilon > 0$ be given. Then,

$$\begin{aligned} &\left|\delta(x_{n_k^{(j)}}, \frac{1}{j}) - \delta(\frac{1}{j}, \frac{1}{j})\right| < 0, \ \forall\ k \in \mathbb{N} \\ \Rightarrow &\lim_{k\to\infty} \delta(x_{n_k^{(j)}}, \frac{1}{j}) = \delta(\frac{1}{j}, \frac{1}{j}) \\ \Rightarrow &\lim_{k\to\infty} x_{n_k^{(j)}} = \frac{1}{j}. \end{aligned}$$

This implies, $\frac{1}{j} \in \Lambda(x)$ and hence $\frac{1}{j} \in \Gamma(x)$. Again for the given $\epsilon > 0$, there exists $j_o \in \mathbb{N}$ such that $\frac{1}{j_o} < \epsilon$. Therefore

$$\begin{aligned} &\{n \in \mathbb{N} : |\delta(x_n, 0) - \delta(0,0)| < \epsilon\} = \bigcup_{j=j_o}^{\infty} \mathcal{D}_j \\ \Rightarrow &d(\{n \in \mathbb{N} : |\delta(x_n, 0) - \delta(0,0)| < \epsilon\}) = \sum_{j=j_o}^{\infty} \frac{1}{2^j} \\ \Rightarrow &d(\{n \in \mathbb{N} : |\delta(x_n, 0) - \delta(0,0)| < \epsilon\}) = \frac{1}{2^{j_o-1}} \neq 0. \end{aligned}$$

This implies that $0 \in \Gamma(x)$. Thus $\Gamma(x) = \{\frac{1}{j} : j \in \mathbb{N}\} \cup \{0\}$. Now let, $\{x_{n_p}\}_{p\in\mathbb{N}}$ be a subsequence of $\{x_n\}_{n\in\mathbb{N}}$ such that $\lim_{p\to\infty} x_{n_p} = 0$ and set $\mathcal{M} = \{n_1, n_2, n_3, \ldots\}$.

Fix $j \in \mathbb{N}$. Then there exists $p_o \in \mathbb{N}$ so that

$$\begin{aligned} &\left|\delta(x_{n_p}, 0) - \delta(0,0)\right| < \frac{1}{j}, \ \forall\ p \geq p_o \\ \Rightarrow &x_{n_p} < \frac{1}{j}, \ \forall\ p \geq p_o. \end{aligned}$$

Therefore, for all $n \in \mathbb{N}$ *with* $n \geq n_{p_o}$ *we have*

$$\begin{aligned}\mathcal{M}(n) &= |\{n_p \in \mathcal{M} : n_p \leq n\}| \\ &= \left|\{n_p \in \mathcal{M} : n_p \leq n, x_{n_p} \geq 1/j\}\right| + \left|\{n_p \in \mathcal{M} : n_p \leq n, x_{n_p} < 1/j\}\right| \\ &\leq (p_o - 1) + \left|\{n_p \in \mathcal{M} : x_{n_p} < 1/j\}\right| \\ \Rightarrow \frac{\mathcal{M}(n)}{n} &\leq \frac{p_o - 1}{n} + \frac{\left|\{n_p \in \mathcal{M} : x_{n_p} < 1/j\}\right|}{n} \\ \Rightarrow \lim_{n\to\infty} \frac{\mathcal{M}(n)}{n} &\leq \frac{1}{2^j} \text{ i.e. } d(\mathcal{M}) \leq \frac{1}{2^j}.\end{aligned}$$

Now taking limit as $j \to \infty$*, we get* $d(\mathcal{M}) = 0$*. Thus any subsequence* $\{x_{n_p}\}_{p\in\mathbb{N}}$ *of* $\{x_n\}_{n\in\mathbb{N}}$ *converging to 0, is a thin subsequence. So,* $0 \notin \Lambda(x)$*. Therefore* $\Lambda(x) \subsetneq \Gamma(x)$*.*

Theorem 3.5. *Let us assume* $x = \{x_n\}_{n\in\mathbb{N}}$ *and* $y = \{y_n\}_{n\in\mathbb{N}}$ *be two sequences of points in a metric-like space* $(\mathcal{Y}, \delta)$ *such that* $d(\{n \in \mathbb{N} : x_n \neq y_n\}) = 0$*. Then* $\Lambda(x) = \Lambda(y)$ *and* $\Gamma(x) = \Gamma(y)$*.*

Proof. Let $x_o \in \Lambda(x)$. Then there exists a subsequence $\{x_{n_k}\}_{k\in\mathbb{N}}$ of $\{x_n\}_{n\in\mathbb{N}}$ such that $d(\{n_1 < n_2 < \ldots\}) \neq 0$ and $\lim_{k\to\infty} x_{n_k} = x_o$. Then

$$\lim_{k\to\infty} \delta(x_{n_k}, x_o) = \delta(x_o, x_o) \tag{3.4}$$

Let $A = \{n_1 < n_2 < \ldots\}$ and $B = \{n \in \mathbb{N} : x_n \neq y_n\}$. Then $d(A) \neq 0$, $d(B) = 0$. So, $d(A - B) \neq 0$. Let $A - B = \{n_{k_1} < n_{k_2} < \ldots\}$. Then we have

$$x_{n_{k_p}} = y_{n_{k_p}},\ \forall\, p \in \mathbb{N}. \tag{3.5}$$

Now $\{x_{n_{k_p}}\}_{p\in\mathbb{N}}$ is a subsequence of $\{x_n\}_{n\in\mathbb{N}}$. So, from (3.4) and (3.5) we have

$$\begin{aligned}&\lim_{p\to\infty} \delta(x_{n_{k_p}}, x_o) = \delta(x_o, x_o) \\ \Rightarrow &\lim_{p\to\infty} \delta(y_{n_{k_p}}, x_o) = \delta(x_o, x_o) \\ \Rightarrow &\lim_{p\to\infty} y_{n_{k_p}} = x_o.\end{aligned}$$

Thus we have, $\{y_{n_{k_p}}\}_{p\in\mathbb{N}}$ is a non-thin subsequence of $\{y_n\}_{n\in\mathbb{N}}$, converging to x_o. So, $x_o \in \Lambda(y)$. Therefore $\Lambda(x) \subset \Lambda(y)$. Similarly we can prove that $\Lambda(y) \subset \Lambda(x)$. Hence we have

$$\Lambda(x) = \Lambda(y).$$

Now, let $x_o \in \Gamma(x)$ and $\epsilon > 0$ be given. Then

$$d(\{n \in \mathbb{N} : |\delta(x_n, x_o) - \delta(x_o, x_o)| < \epsilon\}) \neq 0.$$

Let $C = \{n \in \mathbb{N} : |\delta(x_n, x_o) - \delta(x_o, x_o)| < \epsilon\}$. Then $d(C) \neq 0$ and so, $d(C - B) \neq 0$. Then

$$\begin{aligned}&\{n \in \mathbb{N} : |\delta(y_n, x_o) - \delta(x_o, x_o)| < \epsilon\} \supset (C - B) \\ \Rightarrow\ &d(\{n \in \mathbb{N} : |\delta(y_n, x_o) - \delta(x_o, x_o)| < \epsilon\}) \neq 0.\end{aligned}$$

This gives, $x_o \in \Gamma(y)$. Therefore $\Gamma(x) \subset \Gamma(y)$. Similarly we have, $\Gamma(y) \subset \Gamma(x)$. Hence we have,

$$\Gamma(x) = \Gamma(y).$$

□

4. **Conclusion**

The concept of convergence and summability theory has wide applications in many branches of Mathematics. Also the concept of metric-like space, which is a generalization of partial metric space, presently shed light on researchers. Many studies related to fixed point results have been done in this connection. We have established a necessary and sufficient condition for a point to be a limit point of a sequence in a metric-like space. In this study, we introduce the notion of statistical convergence, statistical limit point, statistical cluster point of a sequence in a metric-like space. In this study we also investigate the interrelationship between the set of all statistical limit points, the set of all statistical cluster points, set of all usual limit points of a sequence in a metric-like space. We showed here that the set of all statistical limits is subset of the set of all statistical limit points, the set of all statistical limit points is subset of all statistical cluster points and that of the set of all statistical cluster points is subset of the set of all limit points of a sequence in a metric-like space. But the inclusion may be proper in each of the cases, which are also shown by giving appropriate examples. We have also proved that under which condition the set of all statistical limit points and the set of all statistical cluster points of two different sequences coincide. In future one can study matrix summability methods in metric-like spaces. one can study other summability methods like $\mathcal{I}$-convergence, $\mathcal{I}$-statistical convergence etc. in metric-like spaces. One can introduce and study the concepts of $\mathcal{I}$-limit points and $\mathcal{I}$-cluster points of a sequences of points in a metric-like spaces.

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